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ON BILATERAL DERIVATES AND THE DERIVATIVE[1]

BY

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ABSTRACT. In this paper we prove a new result on the monotonicity of a function in terms of its bilateral derivates, and obtain from it extensions of several existing results on such derivates and the derivative of a function.

Let $f: R \to R$, where R denotes the set of real numbers. If its lower derivate $\underline{D}f > 0$ at a nonmeager set of points, we prove f to be "adequately" increasing in some interval, viz. even the function $f(x) - \alpha x$ is increasing for some $\alpha > 0$. When f is nowhere adequately monotone, it follows that there exists a residual set of points where f has a zero "median" derivate, i.e. either $D_-f \leqslant 0 \leqslant D^-f$ or $D_+f \leqslant 0 \leqslant D^+f$. These results remain valid for functions defined on an arbitrary set $X \subset R$ under a mild continuity hypothesis, e.g. the absence of ordinary discontinuity at the unilateral limit points of X. The last result leads to a new version of A. P. Morse's theorem for median derivates, and this in turn yields an improved version of the Goldowski-Tonelli theorem. We also obtain some necessary and sufficient conditions for a function to be nondecreasing, and extensions of the mean-value theorem and the Denjoy and other properties of a derivative.

If $f: X \to R$, where $X \subset R$, and both the derivates of f are finite at a set of points that is not meager in X, then f is further proved to satisfy the Lipschitz condition on some portion of X. When f has a finite derivate almost everywhere and X has a finite measure, it is shown that f can be made Lipschitz by altering its values on a set with arbitrarily small measure. Some results on singular functions are also strengthened. The results and the methods of this paper further provide extensions of some results of Young, Tolstoff, Kronrod, Zahorski, Brudno, Fort, Hájek, Filipczak, Neugebauer and Lipiński on derivates and the derivability of a function.

1. Introduction. In this section we first give the nomenclature that is used throughout the work, and then a discussion of the main results that are obtained.

We employ R to denote the set of real numbers, $\bar{R}$ to denote the set of extended real numbers and X to denote an arbitrary but fixed subset of R. Usually

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f denotes a real-valued function with domain X, but in §§4 and 5, unless otherwise stated, the domain of f is assumed to be R.

1.1. DEFINITION. The bilateral lower and upper derivates of a function $f: X \longrightarrow R$ are denoted by $\underline{D}f$ and $\bar{D}f$ respectively, and the unilateral (Dini) derivates of f are denoted as usual by D_-f, D^-f, D_+f and D^+f. We call an extended real number α a *median derivate* of f at $x \in X$ if either $D_-f(x) \leqslant \alpha \leqslant D^-f(x)$ or $D_+f(x) \leqslant \alpha \leqslant D^+f(x)$, and when both the relations hold then α is called a *bilateral median derivate* of f at x. A point $x \in X$ is called in turn a *knot point* of f if $\underline{D}f(x) = -\infty$ and $\bar{D}f(x) = +\infty$, and to be a *bilateral knot point* of f if $D_-f(x) = D_+f(x) = -\infty$ and $D^-f(x) = D^+f(x) = +\infty$.

1.2. DEFINITION. A subset E of X is said to be *countable* if it is finite or countably infinite, and, following Halmos [20], E is called *cocountable* in X if its complement $X - E$ is countable. Further, E is said to be *meager* in X if it is a countable union of sets that are nowhere dense in X, E is *nonmeager* in X if it is not meager in X, and E is *residual* in X if $X - E$ is meager in X. A nonempty subset of X is said to be a *portion* of X if it is of the form $X \cap I$, where I is some open interval in R, and the set X is called *metrically dense-in-itself* [21, p. 192] if it contains in each of its portions a set that has a positive measure.

1.3. DEFINITION. If P is any local property of functions, we call a function $f: X \longrightarrow R$ *cocountably* [or *residually*] P, or P *cocountably* [respectively *residually*] *everywhere*, if the set of points where f possesses P is cocountable [respectively residual] in X. In particular, f is *cocountably* [*residually*] *derivable* if it has a finite or infinite derivative at a cocountable [residual] set of points in X.

1.4. DEFINITION. If P is a local or global property of functions, a function $f: X \longrightarrow R$ is said to be *nowhere* P if it does not possess P on any portion of X, and it is called *intrinsically* P if every portion of X contains a portion on which f possesses P. When P is a local property, an intrinsically P function is also said to be P *intrinsically everywhere*, and f is further said to be P *intrinsically almost everywhere* when every portion of X contains a portion on which f possesses P almost everywhere.

1.5. DEFINITION. Given a function $f: X \longrightarrow R$ and an $\alpha \in R$, we denote by f_α the function

$$f_\alpha(x) = f(x) + \alpha x, \qquad x \in X.$$

Let f be *adequately increasing* [*decreasing*] if there exists a real number $\alpha > 0$ such that the function $f_{-\alpha}$ [f_α] is increasing [decreasing] on X, and let f be *adequately monotone* if it is either adequately increasing or adequately decreasing. The function f is further said to be of *monotonic type* [43, p. 64] if the function f_α is monotone for some $\alpha \in R$.

An adequately increasing function is thus (strictly) increasing, and its derivates are bounded from below by a positive number. The converse of the latter part holds for functions with connected domain [30, p. 266], but not in general. Further, a nowhere monotone function is clearly nowhere adequately monotone as well as nowhere constant. On the other hand, a singular function with a metrically dense-in-itself domain is also nowhere adequately monotone although it may be constant or strictly monotone. A function that is nowhere of monotonic type is also known as a nowhere monotone function of the second species [12].

1.6. DEFINITION. Let a function $f: X \longrightarrow R$ be called *Lipschitz* if it satisfies the Lipschitz condition, and let f be *lower* or *upper Lipschitz* if the function f_α is increasing or decreasing respectively for some $\alpha \in R$. Let, further, f be *lower* or *upper singular* if its derivative is $\geqslant 0$ or $\leqslant 0$ respectively at almost all of the points where it exists, and let f be called *bisingular* if it is lower and upper singular.

A function f is thus Lipschitz if and only if it is lower and upper Lipschitz, and f is of monotonic type if and only if it is lower or upper Lipschitz. Also, f is singular if and only if it is a bisingular function of bounded variation. Since every almost everywhere nonderivable function is bisingular, the class of bisingular functions is much wider. In fact, as it follows from the Denjoy-Young-Saks theorem [23, p. 186], f if bisingular if and only if it has a zero median derivate almost everywhere, and f is lower singular if and only if $\bar{D}f \geqslant 0$ almost everywhere.

We further need some new generalizations of continuity. In analogy to a localized version of the usual notion of regulated functions, we employ the following

1.7. DEFINITION. A function $f: X \longrightarrow R$ is said to be *regulated* at a point $x \in X$ *from the left* [*right*] if the limit $f(x-0)$ [$f(x+0)$] exists whenever x is a limit point of X from the left [right], and f is *regulated* at x if it is so from both the sides of x (i.e. if f does not have a discontinuity of the second kind at x). We call f *interned* at x if it does not have an ordinary discontinuity (i.e. a discontinuity of the first kind) there, or equivalently if f is continuous at x whenever it is regulated there. Let, further, f be interned at x *from the left* or *right* if it does not have an ordinary discontinuity on that side of x, and let f be *bilaterally interned* at x if it is interned there from both the sides.

It may be observed that every function $f: X \longrightarrow R$ is bilaterally interned cocountably everywhere [21, p. 304], and that a Darboux function with a connected domain is always bilaterally interned. The notion "interned" may be compared further with that of "internal" functions in [17]. The following decompositions of the above properties may in turn be compared with lower and upper internal functions there.

1.8. DEFINITION. Let a function $f: X \longrightarrow R$ be *lower* [*upper*] *interned* at $x \in X$ if $f(x-0) \leqslant f(x) \leqslant f(x+0)$ [$f(x-0) \geqslant f(x) \geqslant f(x+0)$] whenever f is regulated at x, where the left or the right inequality is assumed to hold when x is an isolated point of X from the left or right respectively. Let, further, f be lower [upper] interned at x *from the left* or *right* if the left or right inequality holds whenever $f(x - 0)$ or $f(x + 0)$ exists respectively, and let f be *bilaterally lower* [*upper*] *interned* at x if it is lower [upper] interned there from both the sides. As usual, f is said to be (bilaterally) interned, or (bilaterally) lower or upper interned, if it is so at every point of X.

Thus f is (bilaterally) interned at $x \in X$ if and only if it is (bilaterally) lower and upper interned there. It may be further noted that f is bilaterally lower interned whenever it is lower Lipschitz, or, in particular, when f is nondecreasing.

1.9. DEFINITION. Let two points x and y of X be called a *pair of contiguous points* of X if X does not contain any point in between x and y, and let a function f: $X \longrightarrow R$ be *contiguously lower* or *upper interned* if it is lower or upper interned respectively at at least one of each pair of contiguous points of X. We further call f *contiguously interned* if it is contiguously lower and upper interned.

Two points of a closed set X thus form a pair of contiguous points of the set if and only if they are the endpoints of some contiguous (or complementary) interval of X. Also, a function is always contiguously interned when its domain has no contiguous points, e.g. when it is connected or bilaterally dense-in-itself.

We first investigate in §2 some intrinsic properties of functions (see 1.4 for definition) in terms of their bilateral derivates. Let $X \subset R$. A contiguously upper interned function (see 1.9) f: $X \longrightarrow R$ is proved in Theorem 2.3 to be adequately increasing (see 1.5) on some portion of X whenever $\underline{D}f > 0$ at a nonmeager set of points in X. In case f is contiguously interned, it follows that f is of monotonic type (see 1.5) on some portion of X whenever the set of its knot points (see 1.1) is not residual in X. This provides a strengthening of some of the results of Tolstoff [36], Filipczak [6] and Neugebauer [31]. Extending some earlier results on nowhere monotone functions [10], [12], [16], we prove in Theorem 2.9 that if a contiguously interned function f: $X \longrightarrow R$ is nowhere adequately monotone (see 1.4, 1.5), then it has a zero median derivate residually everywhere (see 1.1, 1.3), and if f is nowhere of monotonic type, there exists a residual set of points in X where every extended real number is a median derivate of f.

When a function f: $X \longrightarrow R$ has its derivates finite at a nonmeager set of points in X, it is proved in Theorem 2.13 to be Lipschitz (see 1.6) on some portion of X. This yields an extension of a result of Fort [7] on discontinuous functions. Neither of the Theorems 2.3 and 2.13 remains valid on replacing the nonmeager set in its hypothesis by a set of positive measure (see Remarks 2.8 and 2.15). If $\underline{D}f > 0$ at a set of points which has a positive measure, then according to Proposition 2.18, f is adequately increasing on some nonempty metrically dense-in-itself perfect set in X. When X has a finite measure and the knot points of f form a set of measure zero, we further prove in Theorem 2.20 that f can be made Lipschitz by altering its values on a set with arbitrarily small measure. This result has been proved by Rjazanov [33] for functions of bounded variation defined on a compact interval.

When the domain X of f consists of almost all the points of some interval in R, extending a well-known result on absolutely continuous functions, f is proved in Proposition 2.24 to be nondecreasing if and only if it is lower absolutely continuous and lower singular (see 1.6, 2.23), and in case f is only lower singular, it is further proved in Theorem 2.26 to be nondecreasing on some portion of X whenever the set of points where $\underline{D}f = -\infty$ and $\bar{D}f \geqslant 0$ is not residual in X. This in turn provides refinements of two earlier results on singular functions [15].

In §3 we obtain from Theorem 2.3 and its Lemma 2.2 extensions of some known properties of bilateral derivates to functions defined on an arbitrary set $X \subset R$. Besides the extension, our proofs are unified and much simpler than the original ones. Let f: $X \longrightarrow R$. Extending first a result of W. H. Young [38], the set of points where f has at least one derivate infinite is proved in Theorem 3.1 to be G_δ relative to X. A result of Kronrod [25] follows from this theorem with its extension. For every $\alpha \in \bar{R}$, the set $\{x \in X: \underline{D}f \leqslant \alpha\}$ is proved in Theorem 3.4 to be of the form $G_\delta \cup C$, where the first set is G_δ relative to X and C is a countable set at no point of which f is bilaterally upper interned (see 1.8). This in turn provides improvement of some of the results of Brudno [2], Zahorski [41] and Hájek [19]. A similar result is obtained in Proposition 3.9 for the set of points where f has a median derivate α ($\in \bar{R}$), and this yields a strengthened form of a recent result of Lipiński [27] on singular functions. Theorem 3.12 is an extension of the well-known result of Zahorski [39], [40] and Brudno [2] on the set of points of nonderivability of f.

Let, now, f be a lower interned function (see 1.8) with domain R. In §4 we investigate the nature of derivates of f at the set $N(f)$ of points in no neighborhood of which f is nondecreasing. If $\bar{D}f \geqslant 0$ at a dense set of points in R, f is proved in Theorem 4.7 to have a zero median derivate residually everywhere in $N(f)$. This yields the following property of median derivates of a lower interned function: If f has a median derivate $\geqslant \alpha$ ($\in R$) at a dense set of points and a median derivate $< \alpha$ at some point, then it has a median derivate α at a set of points whose power is c. This property was established by A. P. Morse [29] for the unilateral derivate D^+f of a continuous function. As a consequence of this property we have: If $\bar{D}f \geqslant 0$ at a dense set of points and f has a zero median derivate only at a set of points whose power is $< c$, then f is increasing. Światkowski [35] proved this result for cocountably derivable (see 1.3) Darboux functions for which there exists an $\alpha > 0$ such that f has a derivative $\geq \alpha$ at a dense set of points.

When a lower interned function $f: R \longrightarrow R$ is lower singular, it is proved in Theorem 4.11 to have $\underline{D}f = -\infty$ and $\bar{D}f \geq 0$ residually everywhere in $N(f)$. It follows that a function $f: R \longrightarrow R$ is nondecreasing if and only if it is lower interned and lower singular and the points where $\underline{D}f = -\infty$ and $\bar{D}f \geq 0$ form a set whose power is $< c$. The sufficiency part of this result is the well-known theorem of Goldowski [18] and Tonelli [37] on cocountably derivable continuous functions, which was extended by Zahorski [41] to Darboux functions. In case of bilaterally interned functions (see 1.7), the above two theorems on $N(f)$ follow directly from Theorem 2.9 on nowhere adequately monotone functions (see Remark 4.14).

Finally, we obtain in §5 extensions of some known properties of a derivative. Generalizing a result of Zahorski [41], the derivative of a function $f: X \longrightarrow R$ is proved in Theorem 5.1 to be always in Baire class 1 relative to the set of points where it exists. If $f: R \longrightarrow R$ is a bilaterally interned function with symmetrical derivates, we deduce from Theorem 2.9 the Darboux and the mean-value properties of the median derivates of f (see 5.4), and when f is further cocountably derivable, the usual mean-value theorem holds for f and its derivative possesses the Darboux property relative to the set of points where it exists (see 5.6 and 5.7). In case of a bilaterally interned function that is cocountably derivable and nonangular (see 5.8), Theorem 2.3 leads to the Denjoy property of its derivative relative to the set of points where it exists (see 5.10). A result of Marcus [28] is also extended to the stationary sets of derivatives of interned functions.

Some of the above results were obtained in connection with the investigations of a new generalization of the notion of derivative [17]. The methods of the present paper also lead to new results, and improvements of some of the known ones, on approximate and symmetric derivates which will appear subsequently.

2. **Intrinsic properties in terms of bilateral derivates.** In this section, unless otherwise stated, f is a real-valued function defined on an arbitrary set X of real numbers. For the notation X^* employed at the end of this section see 2.23. The results of the present section remain, in fact, valid if their continuity hypotheses hold only intrinsically everywhere.

2.1. NOTATION. Given a function $f: X \longrightarrow R$ and a natural number k, we denote by $\Delta_k(f)$ the set of points x in X for which $f(t) \leq f(x)$ whenever $t \in X \cap (x - 1/k, x)$ and $f(t) \geq f(x)$ when $t \in X \cap (x, x + 1/k)$.

The following lemma on the set $\Delta_k(f)$ is used throughout the paper (see 1.7

and 1.8 for definitions):

2.2. LEMMA. *If $f: X \longrightarrow R$ and k is any natural number, the set $\Delta_k(f)$ contains all of its bilateral limit points that are in X.*

Moreover, if $x \in X$ is a limit point of $\Delta_k(f)$ from some side, then f is regulated and lower interned at x from that side, and $\Delta_k(f)$ contains x whenever f is upper interned there from the side in question.

PROOF. Let $x \in X$ and Δ denote the set $\Delta_k(f)$ for brevity.

Let, first, x be a bilateral limit point of Δ and $t \in X$. When $x - 1/k < t < x$, there exists a point x_1 in Δ such that $t < x_1 < x$, and so we have $f(t) \leqslant f(x_1) \leqslant f(x)$. In case $x < t < x + 1/k$, there similarly exists a point x_2 in Δ such that $x < x_2 < t$, and we have $f(x) \leqslant f(x_2) \leqslant f(t)$. Hence $x \in \Delta$.

Let, next, x be a unilateral limit point of Δ, say from the left. Then there exists an increasing sequence $\{x_i\}$ of distinct points in Δ such that $x - 1/k < x_i < x$ for every i and x_i converges to x. As $\{f(x_i)\}$ is then a nondecreasing sequence of real numbers bounded from above by $f(x)$, it converges to a limit $\leqslant f(x)$. Moreover, for every $t \in X \cap (x_1, x)$, there exists an i for which $x_i \leqslant t < x_{i+1}$, so that $f(x_i) \leqslant f(t) \leqslant f(x_{i+1})$. Hence $f(x - 0)$ exists and we have

$$f(x-0) = \lim_{i \to \infty} f(x_i) \leqslant f(x),$$

i.e. f is regulated and lower interned at x from the left.

In case f is upper interned at x from the left, we have $f(x) \leqslant f(x - 0)$, whence $f(x) = \lim_{i \to \infty} f(x_i)$. Let, again, $t \in X$. If $x - 1/k < t < x$, we still have $f(t) \leqslant f(x)$ as above. In case $x < t < x + 1/k$, there exists an i_0 for which $t < x_{i_0} + 1/k$, and then for every $i \geqslant i_0$ we have $x_i < t < x_i + 1/k$, so that $f(t) \geqslant f(x_i)$. Thus we have $f(t) \geqslant \lim_{i \to \infty} f(x_i) = f(x)$, and so $x \in \Delta$.

The proof is similar when x is a limit point of Δ from the right.

2.3. THEOREM. *Let a function $f: X \longrightarrow R$ be contiguously upper interned.*

(a) *If $\underline{D}f > 0$ at a nonmeager set of points in X, then f is adequately increasing on some portion of X.*

(b) *In case $\underline{D}f > -\infty$ at a nonmeager set of points in X, then f is lower Lipschitz on some portion of X.*

PROOF. (See 1.5, 1.6, 1.9.) (a) Using the notations in 1.5 and 2.1, it may be easily verified that

$$\{x \in X: \underline{D}f > 0\} \subset \bigcup_{k=1}^{\infty} \Delta_k(f_{-1/k}),$$

and so in the present case there exists a natural number k for which the set $\Delta = \Delta_k(f_{-1/k})$ is dense in some portion X_0 of X. Clearly, X_0 may be assumed to have its diameter $< 1/k$. It would suffice to show the function $f_{-1/k}$ to be nondecreasing on X_0, for then the function $f_{-1/2k}$ would be increasing on X_0.

Let $x, y \in X_0$ and $x < y$. If $X \cap (x, y)$ is nonempty, the interval (x, y) contains a point t of Δ, and since $y - x < 1/k$, then we have

$$f_{-1/k}(x) \leqslant f_{-1/k}(t) \leqslant f_{-1/k}(y).$$

Now suppose that $X \cap (x, y)$ is empty. Then, according to the hypothesis, f is upper interned at one of the points x and y, say at x. Since x is an isolated point of X from the right, f becomes regulated at x whenever $f(x - 0)$ exists, and so f is upper interned at x from the left. The function $f_{-1/k}$ is then also upper interned at x from the left. If x is an isolated point of X, then Δ being dense in X_0, it is clear that $x \in \Delta$, and in case x is a limit point of X, it is a limit point of Δ from the left, whence it follows from 2.2 that $x \in \Delta$. Hence we have once again $f_{-1/k}(x) \leqslant f_{-1/k}(y)$, proving thereby that $f_{-1/k}$ is nondecreasing on X_0.

(b) Since $\{x \in X: \underline{D}f > -\infty\} = \bigcup_{k=1}^{\infty} \{x \in X: \underline{D}f > -k\}$, in this case there exists a natural number k for which $\underline{D}f_k > 0$ at a nonmeager set of points in X. As f_k is also contiguously upper interned, it is increasing by above on some portion of X, and then f is lower Lipschitz on that portion of X. This completes the proof of the theorem.

Following are some immediate consequences of 2.3 (see 1.1–1.6):

2.4. Corollary. *Let a function $f: X \longrightarrow R$ be contiguously upper interned, where X is a G_δ set in R. If $\underline{D}f > 0$ or $\underline{D}f > -\infty$ residually everywhere, then f is intrinsically adequately increasing or intrinsically lower Lipschitz respectively.*

2.5. Corollary. *Let a function $f: X \longrightarrow R$ be contiguously interned. If the set K of knot points of f is not residual in X, then f is of monotonic type on some portion of X, and in case K is meager in X and X is G_δ, then f is intrinsically of monotonic type.*

2.6. Corollary. *Let a function $f: X \longrightarrow R$ be contiguously interned, where X is a G_δ set in R. If f has a derivative α ($\in \bar{R}$) residually everywhere, then the function f_β is intrinsically adequately increasing for every real number $\beta > -\alpha$ and it is intrinsically adequately decreasing for every $\beta < -\alpha$.*

2.7. Corollary. *If a contiguously upper interned function $f: X \longrightarrow R$ is upper singular and X is metrically dense-in-itself, then $\underline{D}f \leqslant 0$ residually everywhere.*

Proof. Let, if possible, $\underline{D}f > 0$ at a nonmeager set of points in X. Then there exists, by 2.3(a), a real number $\alpha > 0$ for which the function $f_{-\alpha}$ is increasing on some portion X_0 of X. But then, according to Lebesgue's theorem [23, p. 122], f has a derivative $\geqslant \alpha > 0$ almost everywhere in X_0, and since X_0 contains a set with positive measure, this contradicts the upper singularity of f.

2.8. REMARKS. (a) The converse of each part of 2.3 holds when X is dense-in-itself and G_δ. The converse of 2.4 or 2.6 holds when X is dense-in-itself, and the converse of either part of 2.5 is valid in general.

(b) In connection with 2.3(a) it may be observed that even if $\underline{D}f > 0$ everywhere, the function f need not be increasing on its entire domain unless the latter is connected [30, p. 266]. If the inequality "$\underline{D}f > 0$" in 2.3(a) is weakened to "$\underline{D}f \geqslant 0$", the function f need not be nondecreasing on any portion of X even if X is connected and f is Lipschitz; for the Köpcke's everywhere derivable nowhere monotone function [24] is Lipschitz [10, footnote 8] and has a zero derivative residually everywhere [10, p. 176]. Also, on replacing the term "at a nonmeager set of points" by "almost everywhere", 2.3(a) does not hold in general for Lipschitz functions with a connected domain; for if $\{E_n\}$ is a decomposition of [0, 1] into a sequence of measurable sets each of which is metrically dense in [0, 1], and if ψ_E denotes the characteristic function of E, then the function

$$f(x) = \int_0^x \sum_{n=1}^{\infty} \frac{1}{n} \psi_{E_n}(x)\, dx, \qquad 0 \leqslant x \leqslant 1,$$

is Lipschitz and has, for each n, a derivative $1/n$ at almost every point of E_n. In case the term "at a nonmeager set of points" in 2.3(a) is replaced only by "at a set of points whose measure is positive", then f need not be even nondecreasing on any portion of X as evidenced again by Köpcke's function [11, p. 666]. Furthermore, on replacing the term "at a nonmeager set of points" by "almost everywhere", 2.3(b) also does not hold in general for absolutely continuous functions with connected domain; for there exists an absolutely continuous function on [0, 1] which has a derivative $-\infty$ at a dense set of points (see Choquet [3, Theorem 26]).

(c) If the set of knot points of f has measure zero instead of being not residual or meager in X, 2.5 also ceases to hold in general; for there exists [8] a continuous nowhere monotone singular function on [0, 1] and such a function is nowhere of monotonic type (see 2.25(b)). The first part of 2.5 strengthens the following result of Tolstoff [36, p. 644]: If a continuous function f with a perfect domain X is derivable at a nonmeager set of points in X, then f has bounded variation on some portion of X. For, a function of monotonic type is clearly a difference of two monotone functions, and so has bounded variation on every bounded portion of its domain that contains its bounds. Since an intrinsically monotonic type function is thus intrinsically cocountably continuous and has a finite derivative intrinsically almost everywhere, and since a function with a connected domain is always contiguously interned, the second part of 2.5 strengthens in turn Theorem 2 of [9] and Theorems 1, 3 and 4 of Filipczak [6] on residual derivability of a function. It also generalizes Theorems 8 and 9 of Neugebauer [31] on measurable smooth functions, for such functions have symmetrical derivates residually everywhere [31, Theorem 10]. When X is metrically dense-in-itself, it further follows from 2.5 that the set of points of derivability of a contiguously interned function $f\colon X \longrightarrow R$ has a positive measure whenever it is nonmeager in X. The converse is false as evidenced again by a continuous nowhere monotone singular function (see 2.27).

The following theorem strengthens two recent results [16, Corollary 5.1, Theorem 7] on nowhere monotone Darboux functions with connected domain (see 1.1, 1.4, 1.5):

2.9. THEOREM. *Let a function $f\colon X \longrightarrow R$ be contiguously interned. If f is nowhere adequately monotone, then it has a zero median derivate residually everywhere.*

In case f is nowhere of monotonic type, then there exists a residual set of points in X where every extended real number is a median derivate of f.

PROOF. There exists, by G. C. Young's theorem [23, p. 182], a countable set C in X such that at every point of $X - C$ we have $D_-f \leqslant D^+f$ and $D_+f \leqslant D^-f$. Thus at each point $x \in X - C$, the two intervals $[D_-f(x), D^-f(x)]$ and $[D_+f(x), D^+f(x)]$ of median derivates of f are intersecting, and so $[\underline{D}f(x), \overline{D}f(x)]$ is the set of median derivates of f at x. Moreover, since X is dense-in-itself in each of the above two cases, the set C is meager in X.

If f is nowhere adequately monotone, then according to 2.3(a) there exists a residual set of points H in X where $\underline{D}f \leqslant 0 \leqslant \overline{D}f$, and then the set $H - C$ is also residual in X and f has a zero median derivate at each of its points. In case f is nowhere of monotonic type, then according to 2.5 the set K of its knot points is residual in X, and then the set $K - C$ is again residual in X and at each of its points $\overline{R}$ is the set of median derivates of f.

2.10. COROLLARY. *Let a function $f\colon X \longrightarrow R$ be contiguously interned.*

(a) *The points where f has a median derivate α ($\in \overline{R}$) form a set that is residual in every portion of X in which it is dense.*

(b) *The knot points of f also form a set that is residual in every portion of X in which it is dense.*

PROOF. Since a function is nowhere adequately monotone whenever it has a zero median derivate at a dense set of points, the part (a) follows for finite α on applying the first part of 2.9 to the function $f_{-\alpha}$, and for infinite α it follows in turn from 2.3(b). Further, since a function is nowhere of monotonic type whenever it has a dense set of knot points, the part (b) follows from the second part of 2.9.

In case of residually derivable functions with the help of 2.6 we get

2.11. COROLLARY. *If X is G_δ and a contiguously interned function $f: X \longrightarrow R$ is residually derivable and nowhere adequately monotone, then the function f_α is intrinsically adequately increasing for every real number $\alpha > 0$ and it is intrinsically adequately decreasing for every $\alpha < 0$.*

2.12. REMARK. The converse of each part of 2.9 holds when X is G_δ. When a contiguously interned function $f: X \longrightarrow R$ is nondecreasing, it follows from 2.10(a) that the set of points where $\underline{D}f = 0$ is residual in every portion of X in which it is dense (see Lipiński [27, Lemma 1]). The part (b) of 2.10 may be compared on the other hand with Theorem 2 of Brudno [2] and with Proposition 3′ of [12].

2.13. THEOREM. *Let E be the set of points where a function $f: X \longrightarrow R$ has at least one derivate infinite. If E is not residual in X, then f is Lipschitz on some portion of X, and in case E is meager in X and X is G_δ, then f is intrinsically Lipschitz.*

PROOF. For every natural number k, let

$$A_k = \Delta_k(f_k) \cap \Delta_k(-f_{-k}).$$

It is easy to verify that $X - E = \bigcup_{k=1}^{\infty} A_k$. If E is not residual in X, then there exists a natural number k such that A_k is dense in some portion X_0 of X. We may further assume X_0 to have a diameter $< 1/k$.

If $x \in X$ is a limit point of A_k from the left, then by 2.2 the functions f_k and $-f_{-k}$ are both lower interned at x from the left. Thus f is interned at x from the left, and so are in turn the functions f_k and $-f_{-k}$. It now follows from 2.2 that $x \in A_k$. The set A_k contains similarly each of its limit points in X from the right. Thus A_k is closed relative to X, and so we have $X_0 \subset A_k$. As the diameter of X_0 is $< 1/k$, both the functions f_k and $-f_{-k}$ are nondecreasing on X_0, and so f is Lipschitz on X_0.

This proves the first part of 2.13, from which the second part follows without difficulty.

M. K. Fort [7] proved that if a function $f: R \longrightarrow R$ is discontinuous at a dense set of points, then the points where f has a finite derivative form a meager set. Since such a function is clearly nowhere Lipschitz, we obtain from 2.13 the following extension of Fort's theorem:

2.14. COROLLARY. *If a function $f: X \longrightarrow R$ is nowhere Lipschitz, then there exists a residual set of points in X where at least one derivate of f is infinite.*

2.15. REMARK. The converse of the first part of 2.13 holds when X is G_δ, and the converse of its second part is always valid. As before, if the set E in 2.13 has measure zero instead of being not residual or meager in X, the result does not hold even for absolutely continuous functions with connected domain; for there exists an absolutely continuous function on [0, 1] which has an infinite derivative at a dense set of points [3, Theorem 26].

The converse of 2.14 also holds when X is G_δ. If a nowhere Lipschitz function $f: X \longrightarrow R$ is continuous at a dense set of points, then according to 2.14 there exists a residual set of points in X where f is continuous but does not have a finite derivative (compare with Fort [7, p. 409]).

Besides a weaker condition for a residually derivable function to be intrinsically Lipschitz, the following proposition gives a curious property of such functions when they are nowhere Lipschitz (see Choquet [3, Theorem 26] for an example):

2.16. PROPOSITION. *Let a contiguously interned function $f: X \longrightarrow R$ be residually derivable and X be G_δ.*

(a) *If f is nowhere Lipschitz, then the function f_α is intrinsically monotone for every real number α.*

(b) *If every portion of X contains a set dense in some portion of X such that f has a bounded median derivate at the points of this set, then f is intrinsically Lipschitz.*

PROOF. (a) Let X_0 be an arbitrary portion of X. According to 2.5, X_0 contains a portion X_1 on which f is lower or upper Lipschitz, say lower Lipschitz. Since f is nowhere Lipschitz, there exists by 2.14 a residual set of points in X_1 where at least one derivate of f is infinite, and since $\underline{D}f$ is bounded from below on X_1, we have in fact $\bar{D}f = +\infty$ residually everywhere in X_1. It now follows from the hypothesis that f has a derivative $+\infty$ residually everywhere in X_1, and so, by 2.6, the function f_α is intrinsically increasing on X_1 for every $\alpha \in R$.

(b) Suppose there exists a portion X_0 of X on no portion of which f is Lipschitz. According to the above proof, X_0 contains a portion X_1 on which either f_α is intrinsically increasing for every $\alpha \in R$, or it is intrinsically decreasing for every $\alpha \in R$, say the former. By hypothesis there exists a portion X_2 of X_1 and a real number $\beta > 0$ such that f has a median derivate $\leqslant \beta$ at a dense set of points in X_2. But then $\underline{D}f_{-2\beta} = \underline{D}f - 2\beta \leqslant \beta - 2\beta < 0$ at a dense set of points in X_2, so that $f_{-2\beta}$ cannot be increasing on any portion of X_2, a contradiction.

Although the converse of 2.16(b) is trivially valid, the converse of 2.16(a) does not hold in general as is evident from a constant function.

Next, we investigate the properties of a function for which one of the conditions of 2.3 holds at a set of points whose measure is positive.

2.17. DEFINITION. Let a function $f: X \longrightarrow R$ be called *piecewise adequately increasing* if it has an extension $g: R \longrightarrow R$ for which there exists a partition of R into finitely many intervals such that g is adequately increasing on the interior of every interval of the partition.

2.18. PROPOSITION. *Let $f: X \longrightarrow R$, where X is a measurable set in R with a finite measure. If $\underline{D}f > 0$ almost everywhere, then for every $\epsilon > 0$ there exists a piecewise adequately increasing function $g: X \longrightarrow R$ such that*

$$m\{x \in X: f(x) \neq g(x)\} < \epsilon.$$

PROOF. Let E be the set of points in X where $\underline{D}f > 0$, and let, for every natural number k, $E_k = E \cap \Delta_k(f_{-1/k})$. Then E_k increases to E as k tends to ∞, where $m(X - E) = 0$, and so E is measurable. Also, since E_k contains by 2.2 all of its bilateral limit points that are in E, it is of the form $E \cap (F - F_\sigma) = E \cap G_\delta$, and so is measurable. Hence, given $\epsilon > 0$, there exists a natural number k such that $m(E - E_k) < \epsilon/2$. There further exists a compact set $K \subset E_k$ such that $m(E_k - K) < \epsilon/2$, and then we have

$$m(X - K) = m(X - E) + m(E - E_k) + m(E_k - K) < \epsilon.$$

If $x, y \in K$, $x < y$ and $y - x < 1/k$, we clearly have $f_{-1/k}(x) \leqslant f_{-1/k}(y)$. The function $f_{-1/k}$ is thus nondecreasing on every portion of K whose finite contiguous intervals are all in length $< 1/k$. It is easy to extend $f_{-1/k}/K$ to a piecewise nondecreasing function g on R, and then the restriction of $g_{1/k}$ to X is piecewise adequately increasing and it coincides with f on K.

2.19. COROLLARY. *If a function $f: X \longrightarrow R$ has $\underline{D}f > 0$ at a measurable set of points with a positive measure, then there exists a nonempty metrically dense-in-itself perfect set $P \subset X$ on which f is adequately increasing.*

PROOF. According to the hypothesis there exists a measurable set $E \subset X$ such that $0 < m(E) < \infty$ and $\underline{D}f > 0$ at every point of E. Putting $g = f/E$, it is clear that $\underline{D}g > 0$ at every point of E, and so by 2.18 there exists a set $E_0 \subset E$ with a positive measure such that g is adequately increasing on E_0. Now, according to a theorem of Luzin [21, p. 192], there exists a metrically dense-in-itself perfect set $P \subset E_0$ such that $m(E_0 - P) < \frac{1}{2}m(E_0)$. Then $m(P) > 0$ and f is adequately increasing on P.

The following theorem has been proved by Rjazanov [33] for functions of bounded variation with a compact connected domain:

2.20. THEOREM. *Let $f: X \longrightarrow R$, where X is a measurable set in R with a finite measure. If the knot points of f form a set of measure zero, then for every $\epsilon > 0$ there exists a Lipschitz function $g: X \longrightarrow R$ such that*

$$m\{x \in X: f(x) \neq g(x)\} < \epsilon.$$

PROOF. Let A be the set of points in X where f does not have any infinite bilateral derivate. Then with the help of the Denjoy-Young-Saks theorem, we have $m(X - A) = 0$. Further, for every natural number k, let

$$A_k = \Delta_k(f_k) \cap \Delta_k(-f_{-k}). \tag{1}$$

According to 2.2, A_k is a G_δ set relative to X, and so is measurable. Also A_k increases to the measurable set A as k tends to ∞. Hence, given $\epsilon > 0$, there exists a natural number k such that $m(A - A_k) < \epsilon/2$. There further exists a compact set $K \subset A_k$ such that $m(A_k - K) < \epsilon/2$, and then we have $m(X - K) < \epsilon$.

Let $g(x) = f(x)$ at every point $x \in K$ and let g be defined linearly on the bounded closed contiguous intervals of K and be constant on the two unbounded closed contiguous intervals of K. Let $\mathfrak{I} = \{[a_i, b_i]: i = 1, 2, \ldots, n\}$ be the set of those bounded closed contiguous intervals of K whose lengths are $\geqslant 1/k$ (which is clearly finite), and let

$$M = \max\left\{k, \left|\frac{g(b_i) - g(a_i)}{b_i - a_i}\right|: 1 \leqslant i \leqslant n\right\}. \tag{2}$$

We wish to show that

$$|g(y) - g(x)| \leqslant M|y - x| \tag{3}$$

for every pair of points x, y in R. Let $(x, y) \in \rho$ if and only if (3) holds. It can be easily verified that ρ is an equivalence relation. It would, therefore, suffice to show that $(x, y) \in \rho$ whenever $x < y$ and $y - x < 1/k$. When $x, y \in K$ ($\subset A_k$), we have, according to (1), $f_k(x) \leqslant f_k(y)$ and $-f_{-k}(x) \leqslant -f_{-k}(y)$, whence

$$-k(y - x) \leqslant g(y) - g(x) \leqslant k(y - x),$$

and so $(x, y) \in \rho$. If x, y are in one and the same closed contiguous interval I of K, the result is trivial when I is unbounded, and otherwise, since g is linear on I, the result follows from (2) when $I \in \mathfrak{I}$ and from the above otherwise (for then $m(I) < 1/k$). When x, y are not in one and the same contiguous interval of K, if

x or y still belongs to some contiguous interval of K, due to the transitivity of ρ it can be replaced by the right or left endpoint of that interval respectively, and then $x, y \in K$, whence the result follows once again from above.

Thus g is a Lipschitz function, and since $m(X-K) < \epsilon$, its restriction to X may be taken as the required function.

2.21. COROLLARY. *If $f: X \longrightarrow R$ and X_0 is a measurable subset of X with a finite measure at no point of which f has a knot point, then for every $\epsilon > 0$ there exists a metrically dense-in-itself perfect set $P \subset X_0$ such that $m(X_0 - P) < \epsilon$ and f is Lipschitz on P.*

PROOF. It is clear that the function $g = f/X_0$ has nowhere a knot point, and so there exists by 2.20 a measurable set $E \subset X_0$ such that $m(X_0 - E) < \epsilon/2$ and g is Lipschitz relative to E. If the measure of E is zero we may take P to be the empty set, otherwise there exists a metrically dense-in-itself perfect set $P \subset E$ such that $m(E - P) < \epsilon/2$. Then $m(X_0 - P) < \epsilon$ and f is Lipschitz on P.

If one of the conditions of 2.3 and 2.13 is satisfied only at an uncountable set of points, we still have the following

2.22. PROPOSITION. *Let $f: X \longrightarrow R$ where X is a Borel set in R. If f has $\underline{D}f > 0$, $\underline{D}f > -\infty$ or both of its bilateral derivates finite at uncountably many points, then there exists a nonempty perfect set $P \subset X$ such that f is adequately increasing, lower Lipschitz or Lipschitz respectively on P.*

PROOF. Suppose first that $\underline{D}f > 0$ at uncountably many points. Then since

$$\{x \in X: \underline{D}f > 0\} \subset \bigcup_{k=1}^{\infty} \Delta_k(f_{-1/k}),$$

there exists a natural number k for which the set $A = \Delta_k(f_{-1/k})$ is uncountable. Since A is G_δ relative to X by 2.2, it is an uncountable Borel set in R, and so it contains by the Alexandrov-Hausdorff theorem [26, p. 447] a nonempty perfect set P. We may clearly assume P to have its diameter $< 1/k$, and then $f_{-1/k}$ is nondecreasing on P, whence f is adequately increasing on P.

If $\underline{D}f > -\infty$ at uncountably many points, then since

$$\{x \in X: \underline{D}f > -\infty\} = \bigcup_{k=1}^{\infty} \{x \in X: \underline{D}f > -k\},$$

there exists a natural number k such that $\underline{D}f_k > 0$ at uncountably many points. Hence, by above, there exists a nonempty perfect set $P \subset X$ on which f_k is adequately increasing, and then f is lower Lipschitz on P.

If, at last, f has both of its bilateral derivates finite at uncountably many points, then since

$$\{x \in X: -\infty < \underline{D}f \leqslant \bar{D}f < +\infty\} \subset \bigcup_{k=1}^{\infty} \{\Delta_k(f_k) \cap \Delta_k(-f_{-k})\},$$

there exists a natural number k for which the set $B = \Delta_k(f_k) \cap \Delta_k(-f_{-k})$ is uncountable. It follows from 2.2 as above that B is a Borel set, and so it contains a nonempty perfect set P with diameter $< 1/k$. It is now easy to verify that f is Lipschitz on P.

Finally, we obtain in this section extensions of some known results on singular functions.

2.23. Definition. Let a function $f: X \longrightarrow R$ be *lower* or *upper absolutely continuous* if for every $\epsilon > 0$ there exists a $\delta > 0$ such that for every finite family $\{[a_i, b_i]: 1 \leqslant i \leqslant n\}$ of nonoverlapping intervals with endpoints in X, whenever $\Sigma_{i=1}^{n}(b_i - a_i) < \delta$, we have

$$\sum_{i=1}^{n} \{f(b_i) - f(a_i)\} > -\epsilon \quad \text{or} \quad < \epsilon \quad \text{respectively.}$$

We further employ X^* to denote a set that consists of almost all the points of some interval in R.

A function f is thus absolutely continuous if and only if it is lower and upper absolutely continuous, and f is further lower absolutely continuous whenever it is lower Lipschitz. The following proposition is an extension of a well-known theorem [34, p. 225] on absolutely continuous functions with connected domain (see 1.6):

2.24. Proposition. *A function $f: X^* \longrightarrow R$ is nondecreasing if and only if it is lower absolutely continuous and lower singular.*

Proof. Necessity of the conditions is obvious. To prove their sufficiency, suppose f is lower absolutely continuous and lower singular but is not nondecreasing. Then there exist two points a, b in X^* such that $a < b$ and $f(a) > f(b)$. Putting $\epsilon = \{f(a) - f(b)\}/(b - a)$ and $g = f_\epsilon$, we have $\epsilon > 0$ and $g(a) = g(b)$.

As the function g is equally lower absolutely continuous, there exists a $\delta > 0$ such that for every finite set of nonoverlapping intervals $\{[a_i, b_i]: 1 \leqslant i \leqslant n\}$ with endpoints in X^*, whenever $\Sigma_{i=1}^{n}(b_i - a_i) < \delta$, we have

$$(1) \qquad \sum_{i=1}^{n} \{g(b_i) - g(a_i)\} > -\frac{\epsilon(b-a)}{4}.$$

It can be easily verified by a standard argument that g is a difference of two nondecreasing functions, and so is derivable almost everywhere. Let E denote the set of points in $X^* \cap (a, b)$ where g has a derivative $\geqslant \epsilon$. According to the hypothesis the set E covers almost all the points of $[a, b]$. Moreover, for every $x \in E$ there exists a decreasing sequence $\{x_k\}$ of points in $X^* \cap (a, b)$ such that x_k converges to x and, for every k, we have

$$g(x_k) - g(x) > \frac{\epsilon}{2}(x_k - x). \tag{2}$$

Putting $I_k(x) = [x, x_k]$ for every k, the family $\{I_k(x): x \in E, k = 1, 2, \ldots\}$ is a Vitali cover of E, and so, by the Vitali covering theorem, there exists a finite family $\{I_{k_i}(x_i): 1 \leqslant i \leqslant n\}$ of mutually disjoint intervals in the above cover such that

$$m\left\{E - \bigcup_{i=1}^{n} I_{k_i}(x_i)\right\} < \min\{\delta, (b-a)/2\}. \tag{3}$$

If $\{[a_i, b_i]: 1 \leqslant i \leqslant n+1\}$ denote the remaining subintervals of $[a, b]$, then

$$\sum_{i=1}^{n+1} (b_i - a_i) = m\left\{[a, b] - \bigcup_{i=1}^{n} I_{k_i}(x_i)\right\} = m\left\{E - \bigcup_{i=1}^{n} I_{k_i}(x_i)\right\} < \delta,$$

and so, by (1) and (2), we have

$$g(b) - g(a) = \sum_{i=1}^{n} \{g(x_{k_i}) - g(x_i)\} + \sum_{i=1}^{n+1} \{g(b_i) - g(a_i)\}$$

$$> \sum_{i=1}^{n} \frac{\epsilon}{2}(x_{k_i} - x_i) - \frac{\epsilon}{4}(b-a) = \frac{\epsilon}{2}\left\{m\left(\bigcup_{i=1}^{n} I_{k_i}(x_i)\right) - \frac{b-a}{2}\right\}.$$

Thus, by (3), we have $g(b) - g(a) > 0$, which is a contradiction.

Following are two consequences of 2.24, the first of which is well known for connected X^*, and the second is known [12, p. 85] for continuous singular functions (see 1.6):

2.25. Corollary. *Let a function $f: X^* \longrightarrow R$ be bisingular.*

(a) *If f is absolutely continuous then it is constant.*

(b) *If f is nowhere monotone then it is nowhere of monotonic type.*

2.26. Theorem. *If a function $f: X^* \longrightarrow R$ is lower singular and the set of points where $\underline{D}f = -\infty$ and $\bar{D}f \geqslant 0$ is not residual in X^*, then f is nondecreasing on some portion of X^*.*

Proof. Since X^* is metrically dense-in-itself and has no contiguous points, we have, by 2.7, $\bar{D}f \geqslant 0$ residually everywhere in X^*. Hence, according to the hypothesis, there exists a nonmeager set of points in X^* where $\underline{D}f > -\infty$. It now follows from 2.3(b) that f is lower Lipschitz on some portion of X^*, and, by 2.24, f is nondecreasing on that portion of X^*.

The above theorem yields the following strengthening of Theorems 1(a) and 2(a) of [15]:

2.27. COROLLARY. *If a lower singular function f: $X^* \longrightarrow R$ is nowhere nondecreasing, then $\underline{D}f = -\infty$ and $\bar{D}f \geqslant 0$ residually everywhere.*

In case f is bisingular and nowhere monotone, then it has a knot point residually everywhere.

2.28. COROLLARY. *Let X^* be G_δ and f: $X^* \longrightarrow R$ be residually derivable. If f is (intrinsically) lower singular, then it is intrinsically nondecreasing, and if f is bisingular then it is intrinsically constant.*

3. **Properties of bilateral derivates and points of derivability.** In this section f continues to be a real-valued function defined on an arbitrary set X in R. A subset of X would be called an F_σ, G_δ, $F_{\sigma\delta}$ or $G_{\delta\sigma}$ set if it is so with respect to the relative topology of X.

The following theorem was proved by W. H. Young [38] for connected X, which was also rediscovered later by Brudno [2]:

3.1. THEOREM (W. H. YOUNG). *The points where a function f: $X \longrightarrow R$ has at least one derivate infinite form a G_δ set (relative to X).*

PROOF. Let A denote the set of points in X where f does not have an infinite derivate. Then a point $x \in A$ is either an isolated point of X or we have $\underline{D}f > -\infty$ and $\bar{D}f < +\infty$ at x. Putting, for every natural number k (see 1.5, 2.1), $A_k = \Delta_k(f_k) \cap \Delta_k(-f_{-k})$, we have $A = \bigcup_{k=1}^\infty A_k$. As we saw in the proof of 2.13, the set A_k is closed relative to X for every natural number k. Hence A is F_σ and the required set is in turn G_δ relative to X.

Following is a theorem of Kronrod [25] (for connected X) for which 3.1 provides a simple proof with an extension:

3.2. COROLLARY (KRONROD). *If the set of points where a function f: $X \longrightarrow R$ is discontinuous is not G_δ, then there exists an uncountable set of points where f is continuous but does not have a finite derivative, and this set has power c when X is a Borel set.*

PROOF. Let D denote the set of points in X where f is discontinuous, and E be the set of points where at least one derivate of f is infinite. Then $D \subset E$, where D is F_σ and E is G_δ. If $E - D$ is countable, then it is an F_σ set, and so we have $D = E - (E - D) \in G_\delta$, contrary to the hypothesis. Hence $E - D$ is uncountable. When X is a Borel set in R, since $E - D = G_\delta - F_\sigma$ is G_δ relative to X, $E - D$ is also a Borel subset of R, and so it has power c by Souslin's theorem [26, p. 479].

3.3. REMARK. The result 3.2 ceases to hold if the term "finite" is deleted from it. For there exists [9, Theorem 1′] an everywhere derivable function on [0, 1] whose points of discontinuity form a countable dense set, and since this set is meager, it cannot be G_δ. Also, if a function has a finite derivative at each of its points of continuity, it follows from 3.2 that its points of discontinuity form an ambiguous set of class 1 (see Kronrod [25]).

3.4. THEOREM. *If $f: X \longrightarrow R$ and $\alpha \in \bar{R}$, then the set $\{x \in X: \underline{D}f \leqslant \alpha\}$ is of the form $G_\delta \cup C$, where C is a countable set at no point of which f is bilaterally upper interned.*

PROOF. (See 1.8.) Let X_0 denote the set of isolated points of X, E be the set of points in X where f is not bilaterally upper interned, and let, for every $\alpha \in R$, $H_\alpha = \{x \in X: \underline{D}f \leqslant \alpha\}$.

Considering first $\alpha = 0$, we have

$$X - H_0 = X_0 \cup \{x \in X: \underline{D}f > 0\} = \bigcup_{k=1}^{\infty} \Delta_k(f_{-1/k}).$$

For every natural number k, the set $\Delta_k(f_{-1/k})$ contains, according to 2.2, all of its bilateral limit points in X and those of its unilateral limit points which are in $X - E$. Thus $\Delta_k(f_{-1/k}) = F_k - C_k$, where F_k is closed relative to X and C_k is a countable subset of E. Hence

$$X - H_0 = \bigcup_{k=1}^{\infty} (F_k - C_k) = F_\sigma - C,$$

where the first set is F_σ relative to X and $C \subset \bigcup_{k=1}^{\infty} C_k$, whence C is again a countable subset of E. This proves the result for $\alpha = 0$.

For a general $\alpha \in R$ the result follows on applying the above to the function $f_{-\alpha}$. For $\alpha = +\infty$ we have $H_{+\infty} = X - X_0$, and this is clearly G_δ since X_0 is countable. For $\alpha = -\infty$ we have $H_{-\infty} = \bigcap_{k=1}^{\infty} H_{-k}$, and the result follows from that on H_{-k} for each k.

As a consequence of 3.4 we obtain the following result of Hájek [19] who proved it for connected X:

3.5. COROLLARY (HÁJEK). *For every function $f: X \longrightarrow R$, its bilateral derivates $\underline{D}f$ and $\bar{D}f$ are in Baire class 2 relative to the set of points where they are defined.*

PROOF. Let E denote the set of nonisolated points of X. For every $\alpha \in R$, the set $\{x \in E: \underline{D}f \leqslant \alpha\}$ is by 3.4 simultaneously $F_{\sigma\delta}$ and $G_{\delta\sigma}$ relative to E, and so the sets

$$\{x \in E: \underline{D}f > \alpha\} \quad \text{and} \quad \{x \in E: \underline{D}f < \alpha\} = \bigcup_{k=1}^{\infty} \{x \in E: \underline{D}f \leqslant \alpha - 1/k\}$$

are both $G_{\delta\sigma}$ relative to E. Since every open set in $\bar{R}$ is a countable union of intervals of the form $(\alpha, +\infty]$ and $[-\infty, \alpha)$, $\alpha \in R$, or their intersections, it follows from the Lebesgue-Hausdorff theorem [26, p. 393] that $\underline{D}f$ is in Baire class 2. On applying this to the function $-f$ one obtains the result for $\bar{D}f$.

It also follows from 3.4 that if a subset X_0 of X contains a dense set of points where f is bilaterally upper interned and $\underline{D}f \leqslant \alpha$ $(\in \bar{R})$, or if X_0 contains in each of its portions uncountably many points where $\underline{D}f \leqslant \alpha$, then $\underline{D}f \leqslant \alpha$ residually everywhere in X_0. Furthermore, since f is bilaterally upper interned at every point where $\bar{D}f < +\infty$, we get

3.6. Corollary. *If f: $X \longrightarrow R$, $\alpha \in \bar{R}$ and a subset X_0 of X contains a dense set of points where $\bar{D}f \leqslant \alpha$, then $\underline{D}f \leqslant \alpha$ residually everywhere in X_0.*

3.7. Corollary. *The knot points of a function f: $X \longrightarrow R$ form a set of the form $G_\delta \cup C$, where C is a countable set at no point of which f is bilaterally interned.*

3.8. Remark. The results 3.4 and 3.7 were obtained by Brudno [2, Lemma 4, Theorem 2] for connected X except for the nature of f at the points of C. According to 3.4 the set $\{x \in X: \underline{D}f \leqslant \alpha\}$ becomes G_δ whenever f is bilaterally upper interned. This in turn has been proved by Zahorski [41, Lemma 8] for cocountably derivable Darboux functions with connected domain.

3.9. Proposition. *If f: $X \longrightarrow R$ and $\alpha \in \bar{R}$, the set E of points where f has a median derivate α is of the form $G_\delta \cup C$, where C is a countable set at no point of which f is bilaterally interned.*

Moreover, the set E is residual in X whenever f has a derivative α at a dense set of points in X.

Proof. Let A be the set of points in X where $\underline{D}f \leqslant \alpha \leqslant \bar{D}f$. Clearly, E is a subset of A. If $x \in A - E$, since f does not have a median derivate α at x, x is a bilateral limit point of X and either $D^{-}f < \alpha < D_{+}f$ or $D^{+}f < \alpha < D_{-}f$ at x. The set $A - E$ is therefore countable by G. C. Young's theorem. According to 3.4 the set A is clearly of the form $G_\delta \cup C$, where C is a countable set at no point of which f is bilaterally interned. Hence

$$E = A - (A - E) = G_\delta \cup C - F_\sigma = G_\delta \cup C_0, \quad \text{where } C_0 \subset C.$$

In case f has a derivative α at a dense set of points in X, the set A is residual in X by 3.6, and since X is presently dense-in-itself, the countable set $A - E$ is meager in X, whence $E = A - (A - E)$ is residual in X.

As a bisingular function (see 1.6) has a zero median derivate almost everywhere [23, p. 186], Proposition 3.9 provides the following strengthened form of Theorems (*) and 1 of Lipiński [27] on singular functions:

3.10. COROLLARY. *Let $f: X \longrightarrow R$ be a bisingular function, where X is metrically dense-in-itself. Then there exists a G_δ set of points in X covering almost all the points of X where f has a zero median derivate. The points where f has a nonzero derivative form in turn a meager set that is contained in an F_σ set of measure zero.*

For the converse of the latter part see Lipiński [27, Theorem 2].

In connection with 3.6, we further obtain from 2.3(a) the following result of Filipczak [6, p. 74] who proved it for connected X (see 1.9):

3.11. PROPOSITION (FILIPCZAK). *If $f: X \longrightarrow R$ is contiguously upper interned, then the set of points where $\underline{D}f \leqslant \alpha$ $(\in \overline{R})$ is residual in every portion of X in which it is dense.*

PROOF. Let H_α denote the set of points in X where $\underline{D}f \leqslant \alpha$, and let us first consider $\alpha = 0$. Suppose H_0 is dense in some portion X_0 of X without being residual in X_0. Then by 2.3(a) there exists a real number $\beta > 0$ for which the function $f_{-\beta}$ is increasing on some portion X_1 of X_0. But then $\underline{D}f_{-\beta} \geqslant 0$, i.e. $\underline{D}f \geqslant \beta > 0$, at every point of X_1, and this is contrary to the hypothesis. This proves the result for $\alpha = 0$.

For a general $\alpha \in R$, the result follows on applying the above to the function $f_{-\alpha}$, which is equally contiguously upper interned. If $H_{+\infty}$ is dense in some portion X_0 of X, then X_0 has no isolated points and we have $X_0 \subset H_{+\infty}$. In case $H_{-\infty}$ is dense in some portion X_0 of X, then so is H_{-k} for every natural number k, and so by above H_{-k} is residual in X_0 for every k, whence the set $H_{-\infty} = \bigcap_{k=1}^{\infty} H_{-k}$ is residual in X_0.

The following theorem was proved originally by Zahorski [39], [40] for continuous functions, and was later extended by Brudno [2] to arbitrary functions with connected domain:

3.12. THEOREM (ZAHORSKI-BRUDNO). *The points where a function $f: X \longrightarrow R$ does not have a derivative, or where f does not have a finite derivative, form a set of the form $G_\delta \cup G_{\delta\sigma}$ where the latter set is of measure zero.*

PROOF. Let N denote the set of points in X where f does not have a derivative. If X_0 is the set of isolated points of X and Q is the set of rational numbers, we have

$$N = X_0 \cup \{x \in X: \underline{D}f < \bar{D}f\}$$
$$= X_0 \cup \bigcup_{\alpha,\beta \in Q;\alpha<\beta} [\{x \in X: \underline{D}f \leqslant \alpha\} \cap \{x \in X: \bar{D}f \geqslant \beta\}].$$

Since X_0 is countable, whence F_σ, it follows from 3.4 that N is $G_{\delta\sigma}$ relative to X.

Further, N contains the set K of knot points of f and according to 3.7, $K = G_\delta \cup C$, where C is countable. Hence $N = K \cup (N - K) = G_\delta \cup H$, where $H = C \cup (N - K) = F_\sigma \cup (G_{\delta\sigma} - F_{\sigma\delta}) \in G_{\delta\sigma}$. Also, by the Denjoy-Young-Saks theorem, the set $N - K$ has measure zero, and so in turn H has measure zero. This proves the result for N.

On the other hand, if N^* denotes the set of points in X where f does not have a finite derivative, and E is the set of points where f has at least one derivate infinite, we have $N^* = N \cup E$, where E is G_δ by 3.1. Hence $N^* = (G_\delta \cup G_{\delta\sigma}) \cup G_\delta = G_\delta \cup G_{\delta\sigma}$, where the latter set has measure zero. Hence the theorem.

3.13. Corollary. *If X is G_δ, then every residually derivable function $f: X \longrightarrow R$ is intrinsically almost everywhere derivable.*

3.14. Remark. It further follows from 3.12 that a function $f: X \longrightarrow R$ is residually derivable whenever the set Δ of its points of derivability is nonmeager in every portion of X, for as Δ is $F_{\sigma\delta}$ relative to X, it possesses the Baire property relative to X [26, p. 88]. Also, Δ is meager in X whenever $X - \Delta$ has a positive outer measure in every portion of X (see Filipczak [6, Theorem 4]).

As proved by Zahorski [39], [40], the converse of 3.12 holds for continuous functions with connected domain. It would be interesting to investigate if the converse remains valid when the domain is an arbitrary perfect set in R.

4. **Monotonicity in terms of bilateral and median derivates.** In this section f is assumed to be a real-valued function with domain R.

4.1. Notation. Given a function f, we denote by $N(f)$ the set of points x in R for which f is not nondecreasing in any neighborhood of x.

4.2. Lemma. (a) *If a function f is lower interned, then the set $N(f)$ is perfect.*

(b) *When f is lower singular, we have $N(f_\alpha) = N(f)$ for every real number $\alpha > 0$.*

Proof. (See 1.6, 1.8.) It is easy to see that the set $N(f)$ is always closed, and that f is nondecreasing in every (open) interval contiguous to $N(f)$.

(a) Let f be lower interned and suppose that $N(f)$ has an isolated point x. Then x is a common endpoint of two contiguous intervals (a, x) and (x, b) of $N(f)$ and f is nondecreasing in each of them. Since f is then regulated at x, according to the hypothesis we have $f(x - 0) \leqslant f(x) \leqslant f(x + 0)$. The function f thus becomes nondecreasing in (a, b), i.e. $x \notin N(f)$, which is a contradiction.

(b) Let f be lower singular and $\alpha > 0$. Since f_α is clearly nondecreasing in every interval in which f is so, we have $N(f_\alpha) \subset N(f)$. Let $x \in N(f)$ and, if possible, $x \notin N(f_\alpha)$. Then there exists an open interval I containing x in which f_α is nondecreasing. The function f is then lower Lipschitz in I, and so is nondecreasing in I by 2.24, i.e. $x \notin N(f)$, a contradiction. Hence the required equality.

4.3. DEFINITION. Let a function f be *conditionally lower* [*upper*] *interned* if it is lower [upper] interned at every point x where f is regulated and $f(x-0) \leqslant [\geqslant] f(x+0)$.

When f is lower or upper interned, it is indeed conditionally lower interned as well as conditionally upper interned.

4.4. THEOREM. *A conditionally lower interned function f has $\underline{D}f \leqslant 0$ residually everywhere in $N(f)$.*

Moreover, if f is further lower singular, then $\underline{D}f = -\infty$ residually everywhere in $N(f)$.

PROOF. Let $N = N(f)$ be nonempty, the result being vacuously true otherwise. Since (see 1.5, 2.1) $\{x\colon \underline{D}f > 0\} = \bigcup_{k=1}^{\infty} \Delta_k(f_{-1/k})$, the first part of the theorem would follow if the set $N \cap \Delta_k(f_{-1/k})$ is nowhere dense in N for every natural number k.

Suppose there exists a k for which $N \cap \Delta_k(f_{-1/k})$ is dense in some portion of N, and let $\Delta = \Delta_k(f_{-1/k})$. Then there exists an open interval I with length $< 1/k$ such that $N \cap I$ is nonempty and $N \cap \Delta$ is dense in $N \cap I$. It would suffice to show f to be nondecreasing in I, for then $N \cap I$ would be empty, which is a contradiction.

Let $x_1, x_2 \in I$ and $x_1 < x_2$. If $N \cap (x_1, x_2)$ is nonempty, the interval (x_1, x_2) contains at least one point of Δ, say x, and then we have

$$f_{-1/k}(x_1) \leqslant f_{-1/k}(x) \leqslant f_{-1/k}(x_2),$$

so that

$$f(x_1) \leqslant f(x_2) - (x_2 - x_1)/k < f(x_2).$$

In case $N \cap (x_1, x_2)$ is empty, the closed set N has a contiguous interval (a, b) such that $a \leqslant x_1 < x_2 \leqslant b$. Let $c = \frac{1}{2}(x_1 + x_2)$. Since f is nondecreasing in (a, b), we have $f(x_1) \leqslant f(c)$ when $x_1 > a$. If $x_1 = a$ and $x_1 \in \Delta$, then since $c - x_1 < 1/k$, we have $f_{-1/k}(c) \geqslant f_{-1/k}(x_1)$, so that $f(c) \geqslant f(x_1) + (c-a)/k > f(x_1)$. In case $x_1 = a \notin \Delta$, then since $N \cap \Delta$ is dense in $N \cap I$, x_1 is the limit of a sequence of points $\{t_i\}$ in Δ from the left. Then by 2.2 the limit $f(x_1 - 0)$

exists, and the limit $f(x_1 + 0)$ already exists since f is nondecreasing in (x_1, b). Given $x \in (x_1, c)$, for all sufficiently large i we have $f(x) > f(t_i)$ as above, and so

$$f(x) \geqslant \lim_{i\to\infty} f(t_i) = f(x_1 - 0),$$

whence $f(x_1 + 0) \geqslant f(x_1 - 0)$. As f is given to be conditionally lower interned, it follows that $f(x_1) \leqslant f(x_1 + 0) \leqslant f(c)$. Thus in every case we have $f(x_1) \leqslant f(c)$, and it can be proved similarly that $f(x_2) \geqslant f(c)$, whence again $f(x_1) \leqslant f(x_2)$. This establishes the first part of the theorem.

Let, now, f be lower singular. Given a natural number k, the function f_k is also conditionally lower interned, and so, by above, the set $N(f_k)$, which is the same as $N(f)$ by 4.2(b), contains a residual subset H_k of points where $\underline{D}f_k \leqslant 0$, i.e. where $\underline{D}f \leqslant -k$. The set $H = \bigcap_{k=1}^{\infty} H_k$ is again residual in $N(f)$, and at every point of H we have $\underline{D}f = -\infty$. This completes the proof of 4.4.

As it can be easily verified, the set $N(f)$ always contains a dense set of points where $\underline{D}f < 0$. In case of lower interned functions we further have

4.5. Corollary. *If a function f is lower interned, then the set of points where $\underline{D}f < 0$ has power c in every portion of $N(f)$.*

Proof. Suppose I is an open interval such that $I \cap N(f)$ is nonempty and the set $\{x \in I \cap N(f): \underline{D}f < 0\}$ has power $< c$. Since $\underline{D}f \geqslant 0$ at every point outside $N(f)$, the power of the set $\{x \in I: \underline{D}f < 0\}$ is then also $< c$. Hence, given $\alpha > 0$, the set $\{x \in I \cap N(f_\alpha): \underline{D}f_\alpha \leqslant 0\}$ has power $< c$. As the function f_α is also lower interned, the set $N(f_\alpha)$ is perfect by 4.2(a), and so the set $\{x \in N(f_\alpha): \underline{D}f_\alpha \leqslant 0\}$ cannot be residual in $N(f_\alpha)$ unless the set $I \cap N(f_\alpha)$ is empty, i.e. unless f_α is nondecreasing in I. Since this holds for every $\alpha > 0$, it follows that f is nondecreasing in I, i.e. $I \cap N(f) = \emptyset$, a contradiction.

Since f is lower interned at every point where $\underline{D}f > -\infty$, from 4.4 and 4.5 we obtain

4.6. Corollary. *A function f is nondecreasing whenever any of the following holds*:

(a) *the points where $\underline{D}f < 0$ form a set with power $< c$ at each of which f is lower interned,*

(b) *f is lower singular and the points where $\underline{D}f = -\infty$ form a set with power $< c$ at each of which f is lower interned.*

Part (a) of the above corollary is known only when $\underline{D}f \geqslant 0$ everywhere [30, p. 266], and part (b) is known when $\underline{D}f > -\infty$ everywhere (see Zahorski [41, p. 18]).

4.7. THEOREM. *If a lower interned function f has $\bar{D}f \geqslant 0$ at a dense set of points, then it has a zero median derivate residually everywhere in $N(f)$.*

PROOF. (See 1.1.) Let $N = N(f)$ be nonempty. According to 4.2(a), N is perfect. If H denotes the set of points in N where f does not have a zero median derivate, we have

$$H = \{x \in N: D^-f < 0 < D_+f\} \cup \{x \in N: D^+f < 0 < D_-f\}$$

$$\cup \{x \in N: \underline{D}f > 0\} \cup \{x \in N: \bar{D}f < 0\}.$$

The first two sets on the right-hand side are countable by G. C. Young's theorem, and so are meager in the perfect set N. The third set is also meager in N by 4.4. It is thus only the last set which needs to be shown to be meager in N. Moreover, since $\{x: \bar{D}f < 0\} = \bigcup_{k=1}^{\infty} \Delta_k(-f_{1/k})$, it would suffice to prove that the set $N \cap \Delta_k(-f_{1/k})$ is nowhere dense in N for every natural number k.

Suppose there exists a k for which $N \cap \Delta_k(-f_{1/k})$ is dense in some portion of N, and let $\Delta = \Delta_k(-f_{1/k})$. Then there exists an open interval I with length $< 1/k$ such that $N \cap I$ is nonempty and $N \cap \Delta$ is dense in $N \cap I$. In case N is dense in some open subinterval I_0 of I, then so is Δ, and since Δ contains all of its bilateral limit points by 2.2, we have $I_0 \subset \Delta$. But then the function $-f_{1/k}$ is nondecreasing on I_0, and so at every point of I_0 we have $\bar{D}f_{1/k} \leqslant 0$, i.e. $\bar{D}f \leqslant -1/k < 0$, contrary to the hypothesis.

When N is, on the other hand, nowhere dense in I, since it is perfect, we can find a contiguous interval (a, b) of N such that $a, b \in I$. Then $f(a + 0)$ exists, and as a is from its left a limit point of N, and so of Δ, according to 2.2 the limit $-f_{1/k}(a - 0)$ exists, whence $f(a - 0)$ exists. Thus f is regulated at a, and so it is lower interned at a from either side. As the function $-f_{1/k}$ is then upper interned at a from the left, it follows from 2.2 that $a \in \Delta$. However, since f is lower interned at a from the right and it is nondecreasing in (a, b), we have $f(a) \leqslant f(a + 0) \leqslant f((a + b)/2)$, so that

$$-f_{1/k}\left(\frac{a+b}{2}\right) = -f\left(\frac{a+b}{2}\right) - \frac{a+b}{2k} < -f(a) - \frac{a}{k} = -f_{1/k}(a),$$

and so $a \notin \Delta$, which is again a contradiction. This completes the proof of 4.7.

As the set $N(f)$ is always perfect for a lower interned function f, on applying 4.7 to the function $f_{-\alpha}$ we get the following version of A. P. Morse's theorem [29, Theorem 1] for median derivates:

4.8. COROLLARY. *If a lower interned function f has a median derivate $\geqslant \alpha$ $(\in R)$ at a dense set of points and a median derivate $< \alpha$ at some point, then f has a median derivate α at a set of points whose power is c.*

In particular, the Morse theorem in itself remains valid for every cocountably derivable (see (1.3) lower interned function. Theorem 4.7 also provides the following strengthened form of Theorem 2 of Światkowski [35] on cocountably derivable Darboux functions, which may be compared further with Theorem 5 of Morse [29]:

4.9. COROLLARY. *If a lower interned function f has $\bar{D}f \geqslant 0$ at a dense set of points and it has a zero median derivate only at a set of points whose power is $< c$, then f is increasing.*

As it is evident from any continuous increasing singular function, the last condition of 4.9 is not necessary for f to be increasing. It, however, leads to the following necessary and sufficient condition for a function to be nondecreasing, strengthening this time Theorem 3 of Światkowski [35]:

4.10. COROLLARY. *A function f is nondecreasing if, and only if,* (a) *f is lower interned,* (b) *$\bar{D}f \geqslant 0$ at a dense set of points and* (c) *for every real number $\alpha < 0$ there exists a $\beta \in [\alpha, 0]$ such that f has a median derivate β only at a set of points whose power is $< c$.*

PROOF. The necessity of the conditions is obvious. As for their sufficiency, there exists by 4.9 a sequence $\{\beta_k\}$ of nonpositive real numbers converging to zero such that the function $f_{-\beta_k}$ is increasing for every k, whence f is nondecreasing.

4.11. THEOREM. *If a lower interned function f is lower singular, then $\underline{D}f = -\infty$ and $\bar{D}f \geqslant 0$ residually everywhere in $N(\mathring{f})$.*

PROOF. According to 4.4 we already have $\underline{D}f = -\infty$ residually everywhere in $N(f)$. Further, since the domain of f has no contiguous points, by 2.7 we have $\bar{D}f \geqslant 0$ residually everywhere, and so it follows from 4.7 that $\bar{D}f \geqslant 0$ residually everywhere in $N(f)$. Hence the theorem.

As proved by Goldowski [18] and Tonelli [37], a cocountably derivable continuous function is nondecreasing whenever it is lower singular. Zahorski [41, Theorem 2] extended this theorem to Darboux functions. However, a nondecreasing function need be neither Darboux nor cocountably derivable. Theorem 4.11 provides the following weaker forms of these conditions which are also necessary:

4.12. COROLLARY. *A function f is nondecreasing if, and only if, it is lower interned and lower singular and the points where $\underline{D}f = -\infty$ and $\bar{D}f \geqslant 0$ form a set whose power is $< c$.*

4.13. COROLLARY. *If an interned function f is bisingular and each of the sets $\{x\colon \underline{D}f = -\infty, \bar{D}f \geqslant 0\}$ and $\{x\colon \underline{D}f \leqslant 0, \bar{D}f = +\infty\}$ has a power $< c$, then f is constant.*

Any interned bisingular function (see 1.6) that is not constant is thus non-derivable at uncountably many points. This was proved by Zahorski [41, p. 21] for continuous singular functions. One may also obtain from 4.12 a condition similar to that of 4.13 for an interned function to be Lipschitz.

4.14. REMARK. In case of bilaterally interned functions, let us indicate here how Theorems 4.7 and 4.11 follow, as mentioned in the introduction, directly from 2.9:

(a) Let first $\bar{D}f \geqslant 0$ at a dense set of points. We claim that f is then nowhere adequately monotone relative to $N(f)$. (It need not be nowhere monotone relative to $N(f)$ as is evident from any continuous decreasing singular function.) Let $N = N(f) \neq \emptyset$ and $g = f/N$. Since f is bilaterally interned, it is clearly nondecreasing on the closure of every contiguous interval of N. If g is adequately increasing on some portion $N \cap I$ of N, then f becomes nondecreasing on I, so that $N \cap I$ becomes empty, which is not possible. Let, if possible, g be adequately decreasing on some portion $N \cap I$ of N. Then there exists an $\alpha > 0$ such that g_α is decreasing on $N \cap I$. If N is dense in some subinterval I_0 of I, we have $I_0 \subset N$, so that $\bar{D}f_\alpha \leqslant 0$, or $\bar{D}f \leqslant -\alpha < 0$ everywhere in I_0, contrary to the hypothesis. If, on the other hand, N is nowhere dense in I, since it is perfect, it has a contiuous interval (a, b) with endpoints in I. But then $g_\alpha(a) > g_\alpha(b)$, so that $f(a) > f(b) + \alpha(b - a) > f(b)$, which is again not possible since f is nondecreasing on $[a, b]$. Since g is also bilaterally interned, it thus has, by 2.9, a zero median derivate residually everywhere in N, and the same holds for f since it is an extension of g.

(b) Let now f be lower singular. Then by 2.7 we have $\bar{D}f \geqslant 0$ residually everywhere. Hence $\bar{D}f \geqslant 0$ residually everywhere in $N(f)$ by above. Also, for every natural number k, f_k is equally bilaterally interned and $\bar{D}f_k \geqslant k > 0$ residually everywhere, whence, by above, $\underline{D}f_k \leqslant 0$ residually everywhere in $N(f_k)$. But $N(f_k) = N(f)$ by 4.2(b), and so $\underline{D}f \leqslant -k$ residually everywhere in $N(f)$. Since residual sets are closed under countable intersection, it follows that $\underline{D}f = -\infty$ and $\bar{D}f \geqslant 0$ residually everywhere in $N(f)$.

In case of Darboux functions in Baire class 1 the last condition of 4.12 can be further weakened, viz. it suffices if the points considered there form an F_σ set which is mapped by f into a set of measure zero. The following theorem is somewhat stronger. Let us recall that a function f is said to satisfy *Banach condition* (T_2) if its level $f^{-1}(y)$ is countable for almost all values of y.

4.15. THEOREM. *If a Darboux function f in Baire class* 1 *is lower singular and satisfies Banach condition* (T_2), *and if the points where* $\underline{D}f = -\infty$ *and* $\bar{D}f \geqslant 0$ *form an* F_σ *set, then f is nondecreasing and continuous.*

PROOF. If E denotes the set of points where $\underline{D}f = -\infty$ and $\bar{D}f \geqslant 0$, we have $E = \bigcup_{k=1}^{\infty} F_k$, where $\{F_k\}$ is a sequence of closed sets. Let the set $N = N(f)$ be nonempty. Since E is residual in N by 4.11, there exists a k for which F_k is dense in some portion of N. Thus there exists an open interval I such that $N \cap I$ is nonempty and F_k is dense in $N \cap I$. Since F_k is closed, we then have $N \cap I \subset F_k \subset E$. Thus f is nonderivable at every point of $N \cap I$, and so its derivative is $\geqslant 0$ at every point of I where it exists.

We claim that f is then nondecreasing in I. Let, if possible, $a, b \in I$, $a < b$ and $f(a) > f(b)$. If A denotes the set of points in $[a, b]$ where f is derivable and its derivative is $\leqslant 0$, then according to a theorem of Bruckner [1, Theorem 2] we have $mf(A) \geqslant f(a) - f(b) > 0$. On the other hand, since f has a zero derivative at every point of A, we have $mf(A) = 0$ [34, p. 272]. Hence f is nondecreasing in I, i.e. $N \cap I = \varnothing$, which is a contradiction.

As the set N is thus empty, the function f is nondecreasing on its entire domain, and, being Darboux, it is further continuous.

Since a function f satisfies Banach condition (T_2) whenever it maps its bilateral knot points (see 1.1) into a set of measure zero [13, p. 188], we get at once

4.16. COROLLARY. *If a Darboux function f in Baire class* 1 *is lower singular and the points where $\underline{D}f = -\infty$ and $\bar{D}f \geqslant 0$ form an F_σ set that is mapped by f into a set of measure zero, then f is nondecreasing and continuous.*

4.17. REMARK. As the set $\{x: \underline{D}f = -\infty, \bar{D}f \geqslant 0\}$ for a Darboux function f is already G_δ by 3.4, it is in fact F_σ if and only if it is resolvable (see [26, pp. 96, 418]). Also, since there exist continuous functions which have everywhere a knot point (see Jarník [22, Satz I]), the Banach condition (T_2) in 4.15 is indispensable.

5. **Properties of derivatives.** In 5.1 and 5.2 the domain of f is an arbitrary subset X of R, but in the rest of the section f is assumed to have R as its domain.

The following theorem was proved originally by Zahorski [41, p. 15] for everywhere derivable functions on R, and in the following form it has been obtained recently by Preiss [32, Theorems 3, 8] for connected X:

5.1. THEOREM. *The derivative of every function $f: X \longrightarrow R$ is in Baire class* 1 *relative to the set of points where it exists.*

PROOF. Let Δ denote the set of points in X where f is derivable, and let $\alpha \in R$. By 3.4 we have

$$\{x \in \Delta: f'(x) \leqslant \alpha\} = \Delta \cap \{x \in X: \underline{D}f \leqslant \alpha\} = \Delta \cap (G_\delta \cup C),$$

where at the points of C the function f is not bilaterally upper interned, and so $\bar{D}f = +\infty$. At the points of $\Delta \cap C$ we thus have $f'(x) = +\infty$, and so $\Delta \cap C$ is empty. Hence the set $\{x \in \Delta: f'(x) > \alpha\}$ is F_σ relative to Δ, and on applying this to the function $-f$ it follows that the set $\{x \in \Delta: f'(x) < \alpha\}$ is equally F_σ relative to Δ. The result now follows from the Lebesgue-Hausdorff theorem as in case of 3.5.

With the help of 3.12 we obtain the following corollary, the last part of which is known for infinite α with connected X [2, Theorem 7]:

5.2. COROLLARY. *If $f: X \longrightarrow R$, $\alpha \in \bar{R}$ and Δ is the set of points where f is derivable, then the set $\{x \in \Delta: f'(x) = \alpha\}$ is G_δ relative to Δ and it is $F_{\sigma\delta}$ relative to X.*

From here onwards f is assumed to have R as its domain. If a bilaterally interned function (see 1.7) is everywhere derivable, the Darboux and the mean-value properties of its derivative can be obtained from 5.1 by standard methods. Theorem 2.9 leads, however, to a more general version of these results. We first prove a lemma which we also need elsewhere (see 4.1):

5.3. LEMMA. *If a bilaterally interned function f is not monotone and it does not have anywhere a relative extremum, then the set $N(f) \cap N(-f)$ is non-empty and perfect and f is nowhere monotone relative to this set.*

PROOF. The set $N = N(f) \cap N(-f)$ is clearly closed. Let $\mathcal{I}_+$ and $\mathcal{I}_-$ denote the families of contiguous intervals of the sets $N(f)$ and $N(-f)$ respectively. Since f is clearly nowhere constant, it is (strictly) increasing in every interval of $\mathcal{I}_+$ and is decreasing in every interval of $\mathcal{I}_-$. The intervals of the family $\mathcal{I} = \mathcal{I}_+ \cup \mathcal{I}_-$ are thus mutually disjoint, and they constitute the contiguous intervals of N.

If N is empty, we have $R \in \mathcal{I}$, i.e. f is monotone in R, contrary to the hypothesis. To prove that N is perfect, it would suffice to show that the intervals of $\mathcal{I}$ are nonabutting. Let, if possible, $\mathcal{I}$ have two abutting intervals (a, b) and (b, c). Since f is bilaterally interned, these intervals cannot be both in $\mathcal{I}_-$, or both in $\mathcal{I}_+$. To be specific, let $(a, b) \in \mathcal{I}_-$ and $(b, c) \in \mathcal{I}_+$. Then f is decreasing in $[a, b]$ and increasing in $[b, c]$, whence f has a relative minimum at b, contrary to the hypothesis.

Suppose, next, that f is nondecreasing on some portion N_0 of N, and let a, b, c be three points of N_0 such that $a < b < c$.. Since f is monotone on the closure of every contiguous interval of N, it is then nondecreasing in the entire interval (a, c), so that $b \notin N$, a contradiction. Similarly f cannot be nonincreasing on any portion of N. Hence the lemma.

5.4. DEFINITION. (a) A function f is said to have *symmetrical derivates* [14] if it has $D_-f = D_+f$ and $D^-f = D^+f$ everywhere.

(b) The median derivates (see 1.1) of a function f would be said to have *Darboux property* if whenever f has a median derivate less than α $(\in R)$ at some point a and a median derivate greater than α at another point b, then it has a median derivate α at some point strictly between a and b.

(c) The median derivates of f would be said to have *mean-value property* if for every pair of points $a, b, a < b$, there exists a point in (a, b) where f has a median derivate equal to $\{f(b) - f(a)\}/(b - a)$.

(d) Let, further, the median derivates of f have a *limit* α $(\in \bar{R})$ at a point x in R if for every neighborhood V of α (relative to $\bar{R}$) there exists a neighborhood U of x such that at every point of $U - \{x\}$ all the median derivates of f are in V.

5.5. THEOREM. *If a bilaterally interned function f has symmetrical derivates, and if f has two derivates $< \alpha$ $(\in R)$ and $> \alpha$ at two points a and b, where $a < b$, then either f has a derivative α at some point of (a, b), or it has a (bilateral) median derivate α at a set of points in (a, b) whose power is c.*

PROOF. To be specific, let f have a derivate $< \alpha$ at a and a derivate $> \alpha$ at b. Since f has symmetrical derivates, then we have $D_+f(a) < \alpha$ and $D^-f(b) > \alpha$, i.e. $D_+f_{-\alpha}(a) < 0 < D^-f_{-\alpha}(b)$. The function $f_{-\alpha}$ is thus not monotone in $[a, b]$, and since $f_{-\alpha}$ is also bilaterally interned, it cannot be monotone in (a, b) either.

Let g be the restriction of $f_{-\alpha}$ to (a, b), and suppose g does not have anywhere a zero derivative. Since g has symmetrical derivates, it then cannot have a point of extremum, for at such a point g would have a zero derivative. Hence, by 5.3, the set $N = N(g) \cap N(-g)$ is nonempty and perfect (relative to (a, b)), and the function $h = g/N$ is nowhere monotone. Since g is monotone on the closure of every contiguous interval of N, if h has a limit at some point of N from some side, then g also has a limit at that point from the same side. The function h is, therefore, also bilaterally interned. Now it follows from 2.9 that h has a zero median derivate at a residual set of points H in N. Clearly, H has power c. If $x \in H$, then g also has a zero median derivate at x, and so f has a median derivate α at x. Moreover, since f has symmetrical derivates, it has a bilateral median derivate α at x. This completes the proof of 5.5.

5.6. COROLLARY. *If a bilaterally interned function f has symmetrical derivates, then its median derivates possess the Darboux and the mean-value properties.*

Moreover, if the median derivates of f have a limit α $(\in \bar{R})$ at any point x in R, then f has a derivative α at x.

PROOF. The Darboux property of the median derivates follows directly from 5.5. To prove their mean-value property it would suffice to prove, as usual, the Rolle's theorem for median derivates. Let $a, b \in R$, $a < b$ and $f(a) = f(b)$. We are required to find a point in (a, b) where f has a zero median derivate, and this is trivial when f is constant in $[a, b]$. In case f is not constant in $[a, b]$, it cannot be monotone in that interval either, and so f has positive and negative derivates at two points c, d of $[a, b]$. If $c = d$, one can replace d by any other point of $[a, b]$ where f has a nonzero derivate, failing which there remains nothing to prove. Now f has, by 5.5, a zero median derivate at some point strictly between c and d, which is clearly in (a, b).

Next, let the median derivates of f have a limit α at some point x in R. Given a neighborhood V of α, then there exists a neighborhood U of x such that at each point of $U - \{x\}$ all the median derivates of f are in V. If $y \in U$, $y \neq x$ and $\beta = \{f(y) - f(x)\}/(y - x)$, there exists by above a point z strictly between x and y where f has a median derivate β, and since $z \in U - \{x\}$, we have $\beta \in V$. Hence f has a derivative α at x.

In case of cocountably derivable functions (see 1.3), a similar argument yields

5.7. COROLLARY. *Let f be a bilaterally interned function that has symmetrical derivates and is cocountably derivable. Then the (usual) mean-value theorem holds for f and its derivative possesses the Darboux property relative to the set of points where it exists. Also, each derivate of f possesses the Darboux and the mean-value properties.*

Moreover, if the derivative of f has a limit α $(\in \bar{R})$ at any point x in R, then f has a derivative α at x.

In particular, the derivative of an everywhere derivable bilaterally interned function f possesses the Darboux and the mean-value properties. This has been known for a long time for continuous f, and has also been obtained recently by Preiss [32, Theorem 6] for bilaterally interned f. The first part of 5.7 has been proved by Zygmund [42, pp. 49, 54] for continuous smooth functions without the condition of cocountable derivability.

When a bilaterally interned function f is everywhere derivable, since its derivative is in Baire class 1, it can be deduced from 4.12 by a standard argument that f' has the *Denjoy property*, viz. for every $\alpha, \beta \in R$, $\alpha < \beta$, the set $\{x: \alpha < f'(x) < \beta\}$ is either empty or has a positive measure. With the help of 2.3(a) we get a more general result as follows.

5.8. DEFINITION. A function f is said to be *nonangular* [17] if it has $D_-f \leqslant D^+f$ and $D_+f \leqslant D^-f$ everywhere.

5.9. THEOREM. *If a bilaterally interned function f is nonangular and cocountably derivable, and if* $\alpha, \beta \in \bar{R}$, $\alpha < \beta$ *and the points where f has a derivative belonging to* (α, β) *form a set of measure zero, then either* $\bar{D}f$ *is everywhere* $\leqslant \alpha$ *of* $\underline{D}f$ *is everywhere* $\geqslant \beta$.

PROOF. In case $\alpha = -\infty$, as the points where f has an infinite derivative always form a set of measure zero [34, p. 270], the derivative of f is $\geqslant \beta$ at almost all of the points where it exists, and so, by 4.12, the function $f_{-\beta}$ is then nondecreasing, i.e. $\underline{D}f$ is everywhere $\geqslant \beta$. Similarly, when $\beta = +\infty$ we have $\bar{D}f \leqslant \alpha$ everywhere. So let α and β be both finite.

Let $N = N(-f_{-\alpha}) \cap N(f_{-\beta})$, which is clearly closed. We wish to show that N is in fact perfect. Let $\mathcal{I}_\alpha$ and $\mathcal{I}_\beta$ denote the families of contiguous intervals of $N(-f_{-\alpha})$ and $N(f_{-\beta})$ respectively. Then $\bar{D}f \leqslant \alpha$ at the points of every interval of $\mathcal{I}_\alpha$ and $\underline{D}f \geqslant \beta$ at the points of every interval of $\mathcal{I}_\beta$. The intervals of the family $\mathcal{I} = \mathcal{I}_\alpha \cup \mathcal{I}_\beta$ are thus mutually disjoint, and they constitute the contiguous intervals of N. Since f is bilaterally interned, the intervals of $\mathcal{I}_\alpha$ are nonabutting, and so are the intervals of $\mathcal{I}_\beta$. On the other hand, if $(a, b) \in \mathcal{I}_\alpha$ and $(b, c) \in \mathcal{I}_\beta$, then $f_{-\alpha}$ is nonincreasing in $[a, b]$ and $f_{-\beta}$ is nondecreasing in $[b, c]$, whence $D^- f(b) \leqslant \alpha < \beta \leqslant D_+ f(b)$, and this is not possible since f is nonangular. Similarly we cannot have (a, b) in $\mathcal{I}_\beta$ and (b, c) in $\mathcal{I}_\alpha$. Hence the intervals of $\mathcal{I}$ are nonabutting, and N is perfect.

It would suffice to prove that N is empty. For then $R \in \mathcal{I}$, and so either $\bar{D}f$ is everywhere $\leqslant \alpha$ or $\underline{D}f$ is everywhere $\geqslant \beta$. So let, if possible, N be nonempty.

We first prove that the function $f_{-\alpha}$ is not adequately increasing on any portion of N. Suppose $f_{-\alpha}$ is, to the contrary, adequately increasing on some portion N_0 of N, and let a, b, c be three points of N_0 such that $a < b < c$. Putting $I = (a, c)$, there exists a real number $\gamma \in (\alpha, \beta)$ such that $f_{-\gamma}$ is increasing on $N \cap I$. If (x, y) is any contiguous interval of N that is contained in I, then we have $f_{-\gamma}(x) < f_{-\gamma}(y)$, so that $f_{-\alpha}(x) < f_{-\alpha}(y)$, and so $f_{-\alpha}$ cannot be nonincreasing in (x, y), whence $(x, y) \in \mathcal{I}_\beta$ and $f_{-\beta}$ is nondecreasing in $[x, y]$. The function $f_{-\gamma}$ is, therefore, increasing on the closure of every contiguous interval of N in I, and so it is increasing in the entire interval I. Thus $\underline{D}f \geqslant \gamma > \alpha$ everywhere in I, whence f has, according to the hypothesis, its derivative $\geqslant \beta$ at almost all of the points in I where it exists. It then follows from 4.12 that the function $f_{-\beta}$ is nondecreasing in I, i.e. $I \in \mathcal{I}_\beta$ and $b \notin N$, which is a contradiction.

The function $g = f_{-\alpha}/N$ is thus not adequately increasing on any portion of N. As g is also bilaterally interned, it follows from 2.3(a) that $\underline{D}g \leqslant 0$ residually everywhere in N, whence $\underline{D}f_{-\alpha} \leqslant \underline{D}g \leqslant 0$ and $\underline{D}f \leqslant \alpha$ residually everywhere in N. By an analogous argument we prove that the function $f_{-\beta}$ is not adequately decreasing on any portion of N and get $\bar{D}f \geqslant \beta$ residually everywhere in N. Thus $\underline{D}f \leqslant \alpha < \beta \leqslant \bar{D}f$ residually everywhere in N, which contradicts the cocountable derivability of f. This completes the proof of 5.9.

5.10. COROLLARY. *If a bilaterally interned function f is nonangular and cocountably derivable, then its derivative possesses the Denjoy property relative to the set of points where it exists.*

Furthermore, each of the unilateral and bilateral derivates of f, and every selection of its median derivates, possesses the Denjoy property.

In particular, the derivative of an everywhere derivable bilaterally interned function always possesses the Denjoy property. This was proved originally by Denjoy [5] for finite derivatives and then by Clarkson [4] for arbitrary derivatives of continuous functions. Preiss [32] has obtained it recently for derivatives of everywhere derivable bilaterally interned functions.

We obtain finally from 4.13 an extension of a theorem of Marcus [28] on the stationary sets of derivatives of continuous functions to those of the derivatives of interned functions (see also Preiss [32] for derivatives of Darboux functions). A set E in R is said to be a *stationary set* of a class C of functions defined on R if every $f \in C$ that is constant on E is constant everywhere.

5.11. PROPOSITION. *A set E in R is a stationary set of derivatives of interned functions if, and only if, the complement of E has zero inner measure.*

PROOF. Let E be a set in R whose complement has zero inner measure, and let f be a derivative of some interned function g such that f is constant on E, say $= \alpha$. As the set $\{x: f(x) = \alpha\}$ is measurable by 5.2, f equals α almost everywhere. The function $g_{-\alpha}$ is thus bisingular, and so is constant by 4.13. Hence $f = g' = \alpha$ everywhere.

When the complement of E has an inner measure > 0, there in fact exists a finite derivative [41] which is constant on E but not everywhere. Hence the proposition.

REFERENCES

1. A. M. Bruckner, *An affirmative answer to a problem of Zahorski, and some consequences*, Michigan Math. J. **13** (1966), 15–26. MR **32** #5814.

2. A. Brudno, *Continuity and differentiability*, Mat. Sb. **13** (55) (1943), 119–134. (Russian) MR **7**, 10.

3. G. Choquet, *Application des propriétés descriptives de la fonction contingent à la théorie des fonctions de variable réelle et à la géométrie différentielle des variétés cartésiennes*, J. Math. Pures Appl. (9) **26** (1947), 115–226 (1948). MR **9**, 419.

4. J. A. Clarkson, *A property of derivatives*, Bull. Amer. Math. Soc. **53** (1947), 124–125. MR **8**, 451.

5. A. Denjoy, *Sur une propriété des fonctions dérivées*, Enseignement Math. **18** (1916), 320–328.

6. F. M. Filipczak, *On the derivative of a discontinuous function*, Colloq. Math. **13** (1964), 73–79. MR **30** #3184.

7. M. K. Fort, Jr., *A theorem concerning functions discontinuous on a dense set*, Amer. Math. Monthly **58** (1951), 408–410.

8. K. M. Garg, *On a function of non-symmetrical differentiability*, Ganita **9** (1958), 65–75. MR **21** #5703.

9. ———, *On the derivability of functions discontinuous at a dense set*, Rev. Math. Pures Appl. **7** (1962), 175–179. MR **26** #2557.

10. ———, *On nowhere monotone functions*. I. *Derivates at a residual set*, Ann. Univ. Sci. Budapest. Eötvös Sect. Math. **5** (1962), 173–177. MR **26** #3839.

11. ———, *On nowhere monotone functions*. II. *Derivates at sets of power c and at sets of positive measure*, Rev. Math. Pures Appl. **7** (1962), 663–671. MR **31** #1339.

12. ———, *On nowhere monotone functions*. III. *Functions of first and second species*, Rev. Math. Pures Appl. **8** (1963), 83–90. MR **27** #1545.

13. ———, *Applications of Denjoy analogue*. III. *Distribution of various typical level sets*, Acta Math. Acad. Sci. Hungar. **14** (1963), 187–195. MR **27** #2590.

14. ———, *On asymmetrical derivates of non-differentiable functions*, Canad. J. Math. **20** (1968), 135–143. MR **36** #3930.

15. ———, *On singular functions*, Rev. Roumaine Math. Pures Appl. **14** (1969), 1441–1452. MR **41** #3688.

16. ———, *Monotonicity, continuity and levels of Darboux functions*, Colloq. Math. **28** (1973), 91–103, 162. MR **48** #2317.

17. ———, *On some new notions of derivative*, Mem. Amer. Math. Soc. (submitted).

18. G. Goldowsky, *Note sur les dérivées exactes*, Mat. Sb. **35** (1928), 35–36.

19. O. Hájek, *Note sur la mesurabilité B de la dérivée supérieure*, Fund. Math. **44** (1957), 238–240. MR **20** #1734.

20. P. R. Halmos, *Lectures on Boolean algebras*, Van Nostrand Math. Studies, no. 1, Van Nostrand, Princeton, N. J., 1963. MR **29** #4713.

21. E. W. Hobson, *The theory of functions of a real variable and the theory of Fourier's series*. Vol. I, Dover, New York, 1958. MR **19**, 1166.

22. V. Jarník, *Über die Differenzierbarkeit steitiger Funktionen*, Fund. Math. **21** (1933), 48–58.

23. R. L. Jeffery, *The theory of functions of a real variable*, 2nd ed., Math. Expositions, no. 6, Univ. of Toronto Press, Toronto, Ont., 1953. MR **13**, 216.

24. A. Köpcke, *Ueber eine durchaus differentiirbare stetige Function mit Oscillationen in jedem Intervalle*, Math. Ann. **34** (1889), 161–171.

25. A. Kronrod, *Sur la structure de l'ensemble des points de discontinuité d'une fonction dérivable en ses points de continuité*, Izv. Akad. Nauk SSSR Ser. Mat. 1939, 569–578. (Russian) MR **1**, 302.

26. K. Kuratowski, *Topology*. Vol. I, new ed., rev. and aug., Academic Press, New York; PWN, Warsaw, 1966. MR **36** #840.

27. J. S. Lipiński, *On derivatives of singular functions*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **20** (1972), 625–628. MR **48** #6335.

28. S. Marcus, *Sur les ensembles stationnaires des fonctions dérivées–finies ou infinies*, Com. Acad. R. P. Romîne **12** (1962), 399–402. (Romanian) MR **27** #5875.

29. A. P. Morse, *Dini derivatives of continuous functions*, Proc. Amer. Math. Soc. **5** (1954), 126–130. MR **15**, 610.

30. I. P. Natanson, *Theory of functions of a real variable*. Vol. I, GITTL, Moscow, 1950; English transl., Ungar, New York, 1955. MR **12**, 598; **16**, 804.

31. C. J. Neugebauer, *Symmetric, continuous, and smooth functions*, Duke Math. J. **31** (1964), 23–31. MR **28** #1263.

32. D. Preiss, *Approximate derivatives and Baire classes*, Czechoslovak Math. J. **21** (96) (1971), 373–382. MR **44** #4158.

33. B. V. Rjazanov, *Functions of class* V_γ, Vestnik Moskov. Univ. Ser. I. Mat. Meh. **23** (1968), no. 6, 36–39. (Russian) MR **40** #5804.

34. S. Saks, *Theory of the integral*, Monografie Mat., vol. 7, PWN, Warsaw, 1937.

35. T. Świątkowski, *On the conditions of monotonicity of functions*, Fund. Math. **59** (1966), 189–201. MR 33 #7472.

36. G. Tolstoff, *Sur quelques propriétés des fonctions approximativement continues*, Mat. Sb. **5** (47)(1939), 637–645. MR 1, 206.

37. L. Tonelli, *Sulle derivate esatte*, Mem. Accad. Sci. Ist. Bologna (8) **8** (1930/31), 13–15.

38. W. H. Young, *On the infinite derivates of a function of a single real variable*, Ark. Mat. Astr. Fys. **1** (1903), 201–204.

39. Z. Zahorski, *Punktmengen, in welchen eine stetige Funktion nicht differenzierbar ist*, Mat. Sb. **9** (51) (1941), 487–510. (Russian) MR 3, 73.

40. ———, *Sur l'ensemble des points de non-dérivabilité d'une fonction continue*, Bull. Soc. Math. France **74** (1946), 147–178. MR 9, 231.

41. ———, *Sur la première dérivée*, Trans. Amer. Math. Soc. 69 (1950), 1–54. MR 12, 247.

42. A. Zygmund, *Smooth functions*, Duke Math. J. **12** (1945), 47–76. MR 7, 60.

43. ———, *Trigonometric series*. Vol. 1, Cambridge Univ. Press, New York and London, 1968. MR 38 #4882.

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